The 7 Most Important Equations for Retirement Income Planning

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My Plan for Next 60 Minutes

• Discuss mathematical ideas that are needed for proper retirement income planning.
• Focus on the “heroes” associated with these ideas or breakthroughs.
• Touch upon some of my own research work.
• Goal: Help popularize.

QUESTION: What Number is More Difficult to Forecast?

1. Total stock market return during next decade.
2. Total years you will live/spend in retirement.

What Does $1 Grow to Over a Decade in the U.S. Stock Market?

$1.80 / $3.25 = 55% Variability

The Longevity VIX: How Random is the Remaining Lifetime of a 65 year-old Retiree?
These numbers and studies do not provide any sense of risk...

New York Times: Obituaries...

Remaining Life for 65 year-old:
- Arithmetic Mean: 19.7 years
- Standard Deviation: 11.0 years
- Variability of: (11)/(19.7) = 56%

Your Money or Your Life? Equally Risky

Remaining Years at Retirement

Source: Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org (Data downloaded in Feb. 2012. Calculations by Minjie Zhang, Feb. 2012)
Takeaway: Don’t confuse two moments

Mortality Modeling: Who Gets the Credit?

Benjamin Gompertz (1779-1865)

• British Demographer and Actuary
• Fellow of the Royal Society
• Never attended university!
• Brother-in-law of M. Montefiore

Longevity in the U.S.

Number of Americans…

| Age 90   | 1,900,000 |
| Age 100  | 53,600    |

Source: Census Bureau 2010 (2012)

The Gompertz Law of Mortality

The death rate increases every single year of your life by approximately 9%. Like clockwork, the grim reaper takes 9% more of your cohort this year, compared to the previous year.
Equation #1

\[ \ln[p] = (1 - e^{\frac{t}{b}}) e^{\frac{x - m}{b}} \]

2.7183….

Equation #1

- You are \( x = 57 \) years old. The modal value of life is \( m = 87.25 \) years, and the dispersion coefficient is \( b = 9.5 \) years.
- What is the probability you will live for \( t = 33 \) more years, to the age of 90?

\[ \ln[p] = -1.29427 \]

\[ e^{\ln[p]} = e^{-1.29427} \]

\[ p = (2.7183)^{-1.29427} \]

\[ p = 0.2741 \]

27.4%

The probability a 57-year-old will live to the age of 90, under the given modal and dispersion value

<table>
<thead>
<tr>
<th>Your Current Age</th>
<th>Probability of Living to 90</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>26.6%</td>
</tr>
<tr>
<td>65</td>
<td>29.0%</td>
</tr>
<tr>
<td>85</td>
<td>57.9%</td>
</tr>
</tbody>
</table>
Planning and Budgeting for an uncertain horizon isn’t easy...

People underestimate mortality risks from likely causes of death and overestimate mortality risks from unlikely causes of death...

Journal of Experimental Psychology, Human Learning and Memory
1978, Vol. 4(6), pg. 551-578

Does this make sense?

“...Overall, the estimated mean life expectancies, across three studies, were between 7.3 to 9.2 years longer when solicited in live-to vs. die-by frame....”

Source: Payne, Sagara, Shu, Appelt & Johnson, 2011
Life Expectancy as a Constructed Belief

There is an important economic role for insurance over the entire lifecycle.

This curve will kill you....
Equation #2: Price Your Legacy

\[ A_x = \sum_{i=0}^{\infty} \frac{(i P_x)(q_{x+i})}{(1 + R)^{(i+1)}} \]

If you *really* want to maximize your legacy, then get some (more) life insurance...

Promoting the Value of Life Insurance: Who Gets the Credit?

- Professor of Insurance, Wharton.
- Founder of the American College
- Promoted the concept of human life value (HLV)
- Consultant to the US Government
- Traveled the world giving lectures on the importance of life insurance.

Solomon S. Huebner (1882-1964)

Mortality Rates During Period of Spanish Flu: Years 1917 to 1919

Now that you appreciate the randomness of human longevity....

What is the longevity of your investment portfolio?

PV of $1 for N periods, discounted at R.

\[ PV = \frac{1}{(1 + R)^N} = \frac{1 - (1 + R)^{-N}}{R} \]

Solve for N – how long will the money last?

Equation #3: Continuous Time

\[ L = \frac{1}{g} \ln \left[ \frac{w/M}{w/M - g} \right] \]
Using the equation

• You have a $300,000 nest egg, growing at 3% and you want to withdraw $30,000 per year.

• What is the (expected) longevity of your portfolio?

\[ L = \frac{1}{3\%} \ln \left( \frac{\$30}{\$300} \right) \]

Plug in the numbers...

\[ L = \frac{1}{3\%} \ln \left( \frac{\$30}{\$300} \right) \]

\[ L = 11.9 \text{ years} \]

But, is a 3% real (inflation adjusted) growth rate realistic?

Longevity of Portfolio in Years

<table>
<thead>
<tr>
<th>Nest Egg at Retirement “M”</th>
<th>@ 1.5%</th>
<th>$200,000</th>
<th>$300,000</th>
<th>$400,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20,000</td>
<td>10.8</td>
<td>17.0</td>
<td>23.8</td>
<td></td>
</tr>
<tr>
<td>$25,000</td>
<td>8.5</td>
<td>13.2</td>
<td>18.3</td>
<td></td>
</tr>
<tr>
<td>$30,000</td>
<td>7.0</td>
<td>10.8</td>
<td>14.9</td>
<td></td>
</tr>
<tr>
<td>$35,000</td>
<td>6.0</td>
<td>9.2</td>
<td>12.5</td>
<td></td>
</tr>
</tbody>
</table>

Who Deserves the Credit?

Leonardo Pisano (Fibonacci) (1170-1250)
Interest Rates, Italy (1200−1400)

Ok, but what about when the rate of return on investment is random?

Real World Retirement & Dual Randomness

\[ PV = \frac{1}{(1 + R_1)} + \frac{1}{(1 + R_2)} \]
\[ + \ldots + \frac{1}{(1 + R_1)(1 + R_2)(1 + R_3)} \]

The length of life \( T \) is random, and the year-over-year investment return \( R \) is random. The present value of your retirement income plan is random.

Lifetime Ruin Probability

<table>
<thead>
<tr>
<th>Spend</th>
<th>$4 per $100</th>
<th>$6 per $100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retire</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Age 65</td>
<td>7.6%</td>
<td>22.1%</td>
</tr>
<tr>
<td>Age 75</td>
<td>2.5%</td>
<td>9.8%</td>
</tr>
</tbody>
</table>

Parameters: \( m = 87.25, b = 9.5, \mu = 8\%, \sigma = 20\% \)

Equation #4

Lifetime Ruin Probability (LRP)

\[ P_x = \]
**Equation #4**

**Lifetime Ruin Probability (LRP)**

\[ P_{\lambda t} = \frac{\partial P}{\partial t} \]

**Equation #4**

**Lifetime Ruin Probability (LRP)**

\[ P_{\lambda t} = \frac{\partial P}{\partial t} + (\mu w - 1) \frac{\partial P}{\partial w} + \frac{1}{2} \sigma^2 w^2 \]

**Equation #4**

**Lifetime Ruin Probability (LRP)**

\[ P_{\lambda t} = \frac{\partial P}{\partial t} + (\mu w - 1) \frac{\partial P}{\partial w} + \frac{1}{2} \sigma^2 w^2 \frac{\partial^2 P}{\partial w^2} \]

Must use numerical techniques to solve this PDE.

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**Retirement Ruin Probability:** Who gets the credit?

- Russian Mathematician
- Parents were communist revolutionaries.
- It is said: "What Euclid did for geometry, Kolmogorov did for probability."
- Awarded Order of Lenin seven (7) times.
- Founded schools for children to study math and sciences.

**Andrey N. Kolmogorov**

(1903-1987)
Is it rational to assume a constant spending rate for life?

Economists and financial practitioners have different ways of “thinking” about spending...

Economists and financial practitioners have different ways of “thinking” about spending...

Optimal Spending to Maximize Utility
Equation #5

Who Gets the Credit?

Irving Fisher (1867-1947)

- Professor of Economics, Yale.
- Created first inflation-indices.
- Inventor, entrepreneur, spokesperson, health advocate.
- Best known for his infamously incorrect forecast of the stock market in 1929.
Irving Fisher (1930)

The Theory of Interest

“The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it otherwise might be.”

“Everyone at some point in his life doubtless changes his degree of impatience for income.”

“He expects to die and he thinks: Instead of piling up for the remote future, why shouldn’t I enjoy myself during the few years that remain.”

Optimal Spending Rates from $100 at age 65

Realistic Investment Assumption: 2.5% Real

<table>
<thead>
<tr>
<th>Pre-Existing Pension Annuity</th>
<th>Increasing Longevity Risk Aversion....</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low (1)</td>
<td>Med. (2)</td>
</tr>
<tr>
<td>$0</td>
<td>6.33%</td>
</tr>
<tr>
<td>$1</td>
<td>6.60%</td>
</tr>
<tr>
<td>$2</td>
<td>7.16%</td>
</tr>
<tr>
<td>$5</td>
<td>8.02%</td>
</tr>
</tbody>
</table>

Note: Assumes 5% Survival to Age 100, 25% Survival to Age 85 and 50% to Age 87. Subjective Discount Rate (p) assumed equivalent to real investment rate.

In the past, most retirees had Defined Benefit pensions to “hedge” longevity risk

You just got hired by one of the 100 largest companies in the U.S.

Did they offer you a D.B. Pension?

<table>
<thead>
<tr>
<th>Year 1985</th>
<th>Year 2002</th>
<th>Year 2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>50</td>
<td>9</td>
</tr>
</tbody>
</table>

The IFID Centre, Calculations January 2015.
Do Markets Like “Frozen” DBs?

![Graph showing cumulative abnormal returns over days before/after notice of termination.]

The Value or Price of a Life Annuity

Equation #6

\[ a_x = \sum_{i=1}^{\infty} \frac{tP_x}{(1 + r)^i} \]

Who Gets the Credit?

Edmond Halley (1656-1742)

- British Astronomer Royal
- Savilian Professor of Geometry at Oxford University
- Mapped earth’s magnetic field
- Isaac Newton’s Principia publisher
- Wrote and published hundreds of papers on astronomy and geophysics, and one paper in 1693 on pricing life annuities!

What Moves Annuity Prices?

By: N. Charupat, M. Kamstra and M.A. Milevsky
SSRN Working Paper #2021579
Journal of Risk and Insurance, September 2016
Model of Optimal Timing

• Our “model” is driven by four (main) inputs:
  – What is a normal level for DIA pricing rates? (r)
  – How long before pricing gets back to normal? (κ)
  – How bumpy will the road to normality be? (σ)
  – What is your general level of risk-aversion? (γ)
• We then “derive” an optimal dynamic policy for the process of DIA-tization.
Opportunity Timing Takeway

- If you are (truly) risk-neutral...wait for rates to get back to normal.
- If you are moderately risk averse...then perhaps stick your toe in the water, soon.
- If you are highly risk averse...then start a process of dollar cost averaging (DCA) into the deferred income annuity (DIA), immediately.
Paul Samuelson (1915-2009)

- Professor of Economics MIT
- Nobel Laureate 1970.
- Economic Advisor to J.F.K.
- Many of his students went on to win Nobel prize.
- Author of most popular textbook in economics.
- Was the first to introduce dynamic programing to asset allocation over the lifecycle.

Who Deserves the Final Credit?

For the 7th equation... Read the book

Final Words

- There is a core body of mathematical knowledge every financial advisor or wealth manager must be aware of – to provide competent and unbiased financial advice.
- This area merges finance, economics, actuarial science, insurance theory and even biology. It is interdisciplinary, but with a firm mathematical core.
- The simple question: Will my money last as long as I do? can lead to some very deep and interesting mathematical ideas.
- I hope to continue to be part of it (for a long time.)